

Infinitely generated Lawson homology groups on some rational projective varieties

Wenchuan Hu

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Abstract

We construct rational projective 4-dimensional varieties with the property that certain Lawson homology groups tensored with \mathbb{Q} are infinite dimensional \mathbb{Q} -vector spaces. More generally, each pair of integers p and k , with $k \geq 0$, $p > 0$, we find a projective variety Y , such that $L_p H_{2p+k}(Y)$ is infinitely generated.

We also construct two singular rational projective 3-dimensional varieties Y and Y' with the same homeomorphism type but different Lawson homology groups, specifically $L_1 H_3(Y)$ is not isomorphic to $L_1 H_3(Y')$ even up to torsion.

1 Introduction

This paper gives examples of singular rational projective 4-dimensional varieties with infinitely generated Lawson homology groups even modulo torsion. This is totally different from the smooth case ([Pe], also [H1]), where it is known that all Lawson homology groups of rational fourfolds are finitely generated.

This paper also gives examples of singular rational projective 3-dimensional varieties with the same homeomorphism type but different Lawson homology groups.

For an algebraic variety X over \mathbb{C} , the **Lawson homology** $L_p H_k(X)$ of p -cycles is defined by

$$L_p H_k(X) := \pi_{k-2p}(\mathcal{Z}_p(X)), \quad k \geq 2p \geq 0$$

where $\mathcal{Z}_p(X)$ is provided with a natural topology. For general background, the reader is referred to the survey paper [L2].

Clemens showed that the **Griffiths group** of 1-cycles (which is defined to be the group of algebraic 1-cycles homologically equivalent to zero modulo 1-cycles algebraically equivalent to zero) may be infinitely generated even modulo the torsion elements for general quintic hypersurfaces in \mathbb{P}^4 (cf. [C]). Friedlander showed that $L_1 H_2(X)$ is exactly the algebraic 1-cycles modulo algebraic equivalence (cf. [F]). Hence the Griffiths group of 1-cycles for X is a subgroup of $L_1 H_2(X)$.

This leads to the following question:

(Q): Can one show that $L_p H_{2p+j}(X)$ is **not** finitely generated for some projective variety X where $j > 0$?

In this paper we shall construct, for any given integers p and $j > 0$, examples of rational varieties X for which $L_p H_{2p+j}(X)$, as an abelian group, is infinitely generated. Thus, we answer affirmatively the above question :

Theorem 1.1 *There exists rational projective variety X with $\dim(X) = 4$ such that $L_1 H_3(X) \otimes \mathbb{Q}$ is **not** a finite dimensional \mathbb{Q} -vector space.*

By using the projective bundle theorem given by Friedlander and Gabber([FG]), we have the following corollary:

Corollary 1.1 *For any $p \geq 1$, there exists projective algebraic variety X such that $L_p H_{2p+1}(X)$ is **not** a finitely generated abelian group.*

More generally, we have

Theorem 1.2 *For integers p and k , with $k \geq 0, p > 0$, we can find a projective variety Y , such that $L_p H_{2p+k}(Y)$ is infinitely generated.*

Remark 1.1 *The smoothness is essential here. Compare Theorem 1.1 with the following result proved by C. Peters.*

Theorem 1.3 ([Pe]) *For any smooth projective variety X over \mathbb{C} with $\mathrm{Ch}_0(X) \otimes \mathbb{Q} \cong \mathbb{Q}$, the natural map $\Phi : L_1 H_*(X) \otimes \mathbb{Q} \rightarrow H_*(X, \mathbb{Q})$ is injective. In particular, $L_1 H_*(X) \otimes \mathbb{Q}$ is a finite dimensional \mathbb{Q} -vector space.*

Any rational variety X (smooth or not) has the property that $\mathrm{Ch}_0(X) \otimes \mathbb{Q} \cong \mathbb{Q}$.

Applying the same construction to hypersurfaces in \mathbb{P}^3 , we obtain the following:

Theorem 1.4 *There exist two rational 3-dimensional projective varieties Y and Y' which are homeomorphic but for which the Lawson homology groups $L_1 H_3(Y, \mathbb{Q})$ and $L_1 H_3(Y', \mathbb{Q})$ are not isomorphic even up to torsion.*

Remark 1.2 *In fact, these varieties in Theorem 1.4 have exactly one isolated singular point.*

2 Lawson Homology

In this section we briefly review the definitions and results used in the next section. Let X be a projective variety of dimension m over \mathbb{C} . The group of p -cycles on X is the free abelian group $\mathcal{Z}_p(X)$ generated by irreducible p -dimensional subvarieties.

Definition 2.1 *The **Lawson homology** $L_p H_k(X)$ of p -cycles on X is defined by*

$$L_p H_k(X) := \pi_{k-2p}(\mathcal{Z}_p(X)), k \geq 2p \geq 0,$$

where $\mathcal{Z}_p(X)$ is provided with a natural, compactly generated topology (cf. [F], [L1], [L2]).

Definition 2.2 *The **Griffiths group** $\mathrm{Griff}_p(X)$ of p -cycles on X is defined by*

$$\mathrm{Griff}_p(X) := \mathcal{Z}_p(X)_{\mathrm{hom}} / \mathcal{Z}_p(X)_{\mathrm{alg}}$$

where $\mathcal{Z}_p(X)_{\mathrm{hom}}$ denotes algebraic p -cycles homologous to zero and $\mathcal{Z}_p(X)_{\mathrm{alg}}$ denotes algebraic p -cycles which are algebraically equivalent to zero.

Remark 2.1 *It was shown by Friedlander that $L_p H_{2p}(X) \cong \mathcal{Z}_p(X) / \mathcal{Z}_p(X)_{\mathrm{alg}}$ (cf. [F]). Hence the Griffiths group $\mathrm{Griff}_p(X)$ is a subgroup of the Lawson homology $L_p H_{2p}(X)$. Therefore, for any projective variety X (its homology groups are finitely generated), $\mathrm{Griff}_p(X)$ is infinitely generated if and only if $L_p H_{2p}(X)$ is.*

Remark 2.2 *For a quasi-projective variety U , $L_p H_k(U)$ is also well-defined and independent of the projective embedding (cf. [Li], [L2]).*

Let $V \subset U$ be a Zariski open subset of a quasi-projective variety U . Set $Z = U - V$. Then we have

Theorem 2.1 ([Li]) *There is a long exact sequence for the pair (U, Z) , i.e.,*

$$\cdots \rightarrow L_p H_k(Z) \rightarrow L_p H_k(U) \rightarrow L_p H_k(V) \rightarrow L_p H_{k-1}(Z) \rightarrow \cdots \quad (1)$$

Remark 2.3 *For any quasi-projective variety U , $L_0 H_k(U) \cong H_k^{BM}(U)$, where $H_k^{BM}(U)$ is the Borel-Moore homology. This follows from the Dold-Thom Theorem [DT].*

As a direct application of this long exact sequence, one has the following results [Li]:

2.1 Let $U = \mathbb{P}^{n+1}$ and $V = \mathbb{P}^{n+1} - \mathbb{P}^n$. By the Complex Suspension Theorem [L1], we have, $L_p H_{2n}(\mathbb{C}^n) = \mathbf{Z}$; $L_p H_k(\mathbb{C}^n) = 0$ for any $k \neq 2n$ and $k \geq 2p \geq 0$.

2.2 Let $U = \mathbb{C}^n$ and $V_{n-1} \subset \mathbb{C}^n$ be a closed algebraic set. Set $V_n = \mathbb{C}^n - V_{n-1}$. Then we have

$$0 \rightarrow L_p H_{2n+1}(V_n) \rightarrow L_p H_{2n}(V_{n-1}) \rightarrow L_p H_{2n}(\mathbb{C}^n) \rightarrow L_p H_{2n}(V_n) \rightarrow L_p H_{2n-1}(V_{n-1}) \rightarrow 0$$

and

$$L_p H_{k+1}(V_n) \cong L_p H_k(V_{n-1}), \quad k \neq 2n, 2n+1.$$

3 An Elementary Construction

Construction Let $X = (f(x_0, \dots, x_{n+1}) = 0)$ be a general hypersurface in \mathbb{P}^{n+1} with degree d , and let $V_n := X - X \cap \{\mathbb{P}^n = (x_0 = 0)\}$ be the affine part, i.e. $V_n \subset \mathbb{C}^{n+1}$. Define $V_{n+1} := \mathbb{C}^{n+1} - V_n$, then V_{n+1} can be viewed as an affine variety in \mathbb{C}^{n+2} defined by $x_{n+2} \cdot f(1, x_1, \dots, x_{n+1}) - 1 = 0$, where $V_n = (f(1, x_1, \dots, x_{n+1}) = 0)$. Denoted by $\overline{V_{n+1}}$ the projective closure of V_{n+1} in \mathbb{P}^{n+2} and set $Z_n = \overline{V_{n+1}} - V_{n+1}$.

We leave the study of this case where $n = 1$, and X is a smooth plane curve, as an exercise.

3.1 Application to the Case $n = 2$

In this subsection, I will show that there exist two rational projective 3-dimensional varieties with the same singular homology groups but different Lawson homology.

The following result proved by Friedlander will be used several times:

Theorem 3.1 (Friedlander [F1]) *Let X be any smooth projective variety of dimension n . Then we have the following isomorphisms*

$$\begin{cases} L_{n-1} H_{2n}(X) \cong \mathbf{Z}, \\ L_{n-1} H_{2n-1}(X) \cong H_{2n-1}(X, \mathbf{Z}), \\ L_{n-1} H_{2n-2}(X) \cong H_{n-1, n-1}(X, \mathbf{Z}) = NS(X) \\ L_{n-1} H_k(X) = 0 \quad \text{for } k > 2n. \end{cases}$$

For a finitely generated abelian group G , we denote by $\text{rk}(G)$ the rank of G .

Let $X \subset \mathbb{P}^3$ be a general surface with degree $d = 4$. Then $V_2 = X - X \cap \mathbb{P}^2$ and $C := X \cap \mathbb{P}^3$ is a smooth curve in \mathbb{P}^2 .

Lemma 3.1 $\text{rk}(L_1H_2(X)) \cong \text{rk}(L_1H_2(V_2)) + 1$; $\text{rk}(L_1H_3(V_2)) = 0$.

Proof. Applying Theorem 2.1 to the pair (X, C) and Theorem 3.1 for X , we get

$$0 \rightarrow L_1H_3(V_2) \rightarrow L_1H_2(C) \rightarrow L_1H_2(X) \rightarrow L_1H_2(V_2) \rightarrow 0.$$

Note that $L_1H_2(C) \cong \mathbf{Z}$ and the map $L_1H_2(C) \rightarrow L_1H_2(X)$ is injective, and so we get $L_1H_3(V_2) = 0$. Therefore, by the above long exact sequence, we have $\text{rk}(L_1H_2(X)) \cong \text{rk}(L_1H_2(V_2)) + 1$. \square

Lemma 3.2 $\text{rk}(L_1H_2(Z_2)) = 1$; $\text{rk}(L_1H_3(Z_2)) = 6$; $\text{rk}(L_1H_4(Z_2)) = 2$.

Proof. Note that $Z_2 = \overline{V}_3 - V_3$ is defined by $(x_4 \cdot f(0, x_1, \dots, x_3) = 0, x_0 = 0)$ in \mathbb{P}^4 . Let $C' = (x_4 = 0) \cap (f(0, x_1, \dots, x_3) = 0)$ in the hyperplane $(x_0 = 0) \subset \mathbb{P}^3$. It is easy to see that $C' \cong C$. Then $Z_2 = \mathbb{P}^2 \cup \Sigma_p(C)$, where $\Sigma_p(C)$ means the joint of C and the point $p = [1 : 0 : \dots : 0]$. By applying Theorem 2.1 to the pair $(Z_2, \Sigma C)$, we get

$$\dots \rightarrow L_1H_3(Z_2 - \Sigma C) \rightarrow L_1H_2(\Sigma C) \rightarrow L_1H_2(Z_2) \rightarrow L_1H_2(Z_2 - \Sigma C) \rightarrow 0.$$

Note that $Z_2 - \Sigma C \cong \mathbb{P}^2 - C$ and $L_1H_3(\mathbb{P}^2 - C) = 0$. Therefore $\text{rk}(L_1H_2(Z_2)) = 1$. Moreover, since $L_1H_4(\mathbb{P}^2 - C) = \mathbf{Z}$ and $L_1H_4(\Sigma C) = \mathbf{Z}$ and $\mathbb{P}^2 \cap \Sigma C = C$ is a curve. The last statement follows. Recall that the Complex Suspension Theorem and Dold-Thom Theorem, we have $L_1H_3(\Sigma C) \cong L_0H_1(C) \cong H_1(C)$. By assumption, C is a plane curve of degree 4. The adjunction formula gives $\text{rk}(H_1(C)) = 6$. The second statement follows. \square

Lemma 3.3 $\text{rk}(L_1H_2(\overline{V}_3)) \leq 1$; $\text{rk}(L_1H_3(\overline{V}_3)) = \text{rk}(L_1H_2(X)) + \text{rk}(L_1H_2(\overline{V}_3)) + 4$.

Proof. Applying Theorem 2.1 to the pair (\overline{V}_3, Z_2) with $p = 1$, we have

$$0 \rightarrow L_1H_3(Z_2) \rightarrow L_1H_3(\overline{V}_3) \rightarrow L_1H_3(V_3) \rightarrow L_1H_2(Z_2) \rightarrow L_1H_2(\overline{V}_3) \rightarrow 0$$

since Lemma 3.1 gives $L_1H_2(V_3) = 0$ and $L_1H_4(V_3) \cong L_1H_3(V_2) = 0$. Hence $\text{rk}(L_1H_2(\overline{V}_3)) \leq 1$. Moreover, we have $\text{rk}(L_1H_3(Z_2)) - \text{rk}(L_1H_3(\overline{V}_3)) + \text{rk}(L_1H_3(V_3)) - \text{rk}(L_1H_2(Z_2)) + \text{rk}(L_1H_2(\overline{V}_3)) = 0$. By Lemma 3.2, we get

$$6 - \text{rk}(L_1H_3(\overline{V}_3)) + (\text{rk}(L_1H_2(X)) - 1) - 1 + \text{rk}(L_1H_2(\overline{V}_3)) = 0$$

\square

Lemma 3.4 $\text{Sing}(\overline{V}_3) \cong \{X \cap (x_0 = 0)\} \cup \{p\} \cong C \cup \{p\}$.

Proof. It follows from a direct computation. By definition,

$$\begin{aligned}
\text{Sing}(\overline{V}_3) &= \left\{ \begin{array}{l} F(x_0, x_1, x_2, x_3, x_4) = 0, \\ dF(x_0, x_1, x_2, x_3, x_4) = 0 \end{array} \right\} \\
&= \left\{ \begin{array}{l} x_0^{d+1} - x_4 \cdot f(x_0, x_1, x_2, x_3) = 0, \\ (d+1)x_0 - x_4 \cdot \frac{\partial f}{\partial x_0} = 0, \\ -x_4 \cdot \frac{\partial f}{\partial x_1} = 0, \\ -x_4 \cdot \frac{\partial f}{\partial x_2} = 0, \\ -x_4 \cdot \frac{\partial f}{\partial x_3} = 0, \\ f(x_0, x_1, x_2, x_3) = 0 \end{array} \right\} \\
&= \left\{ \begin{array}{l} x_0 = 0, \\ x_4 \cdot \frac{\partial f}{\partial x_0} = 0, \\ x_4 \cdot \frac{\partial f}{\partial x_1} = 0, \\ x_4 \cdot \frac{\partial f}{\partial x_2} = 0, \\ x_4 \cdot \frac{\partial f}{\partial x_3} = 0, \\ f(x_0, x_1, x_2, x_3) = 0 \end{array} \right\} \\
&= \left\{ x_0 = x_3 = f(x_0, x_1, x_2) = 0 \right\} \cup \left\{ x_0 = \frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{\partial f}{\partial x_3} = 0 \right\} \\
&\cong \{x_0 = f(x_0, x_1, x_2) = 0\} \cup \{p\} \cong C \cup \{p\}
\end{aligned}$$

since $C = (f = 0)$ is smooth by our assumption. \square

Remark 3.1 Note that p is an isolated singular point and the singularity $C = X \cap (x_0 = 0)$ is of A_n -type. We can resolve the singularity of this part by blowing up twice over the singularity, i.e., by blowing up over the singularity for the first time and then blowing up the singularity of the proper transform of the first blowup. We denote by $\widetilde{\widetilde{V}}_3$ the proper transform of \overline{V}_3 with the exceptional divisor D_1 for the first blowup and $\widetilde{\widetilde{V}}_3$ the proper transform of $\widetilde{\widetilde{V}}_3$ with the exceptional divisor D_2 for the second blowup. Both D_1 and D_2 are isomorphic to a fiber bundle over C with fibre the union of two \mathbb{P}^1 intersecting at exactly one point. See the appendix for the computation of a concrete example.

Now $\widetilde{\widetilde{V}}_3$ has only one singular point, denote by q .

Lemma 3.5 The singular point q in $\widetilde{\widetilde{V}}_3$ can be resolved by one blow up whose exceptional divisor is isomorphic to X .

Proof. It follows from a trivial computation. \square

We denote by W_3 the proper transform of the blow up in the above lemma. Note that W_3 is a **smooth** rational threefold. We have the following property on W_3 :

Proposition 3.1 *For a smooth surface $X \subset \mathbb{P}^3$, the W_3 thus constructed is a smooth rational threefold with a fixed homeomorphic type, i.e., for two smooth surfaces X and X' in \mathbb{P}^3 , the corresponding smooth rational threefolds W_3 and W'_3 are homeomorphic.*

Proof. Note that V_3 is a hypersurface in \mathbb{P}^4 . Let $(f_t(x_0, \dots, x_4) = 0) \subset \mathbb{P}^4$ be a family of hypersurface such that $V_3 = (f_0 = 0)$ is transversal to the hypersurface $H = (x_0 = 0)$. Let Δ be a neighborhood of $t = 0$ such that $(f_t = 0)$ is transversal to H for all $t \in \Delta$. Let $W \subset \mathbb{P}^4 \times \Delta$ be the (analytic) variety defined by $F(x, t) := f_t(x) = 0$. Then we have the following incidence correspondence

$$\begin{array}{ccc} W & \subset & \mathbb{P}^4 \times \Delta \\ \downarrow \pi & & \downarrow \pi \\ \Delta & = & \Delta, \end{array}$$

with $\pi^{-1}(t) \cap W = W^t$.

By Remark 3.1 and Lemma 3.5, we get a smooth variety \widetilde{W} by blowing up twice along 2-dimensional singularity of $Sing(W)$ and once for the remainder 1-dimensional singularity of $Sing(W)$. Denote by E the exceptional divisor of the last step. We claim that the map $\tilde{\pi} : \widetilde{W} \rightarrow \Delta$ is a smooth proper submersion. In fact, let v be a vector field of Δ and let \tilde{v} be a lifting in $\Gamma(\widetilde{W}, T\widetilde{W})$ such that $\tilde{\pi}_*(\tilde{v}) = v$.

Denote by φ_t (resp. $\tilde{\varphi}_t$) the flow determined by v (resp. \tilde{v}). Then $\tilde{\varphi}_t : \widetilde{W}^0 \rightarrow \widetilde{W}^t$ gives the homeomorphism between two fiber of $\tilde{\pi}$ from Ehresmann's Theorem [V]. This implies the result of the proposition. \square

From this proposition, we have the following

Corollary 3.1 *For all smooth surfaces $X \subset \mathbb{P}^3$ of fixed degree, the $\widetilde{\widetilde{V}}_3$ thus constructed has a fixed homeomorphism type.*

Proof. In this proof of the proposition, we actually can choose \tilde{v} such that 1) \tilde{v} is tangent to W ; 2) \tilde{v} is tangent to the exceptional divisor E . Then the flow of \tilde{v} gives the homeomorphism of any two fibers. \square

We want to show that some Lawson homology group of $\widetilde{\widetilde{V}}_3$ may vary when the general X varies in \mathbb{P}^3 .

Theorem 3.2 *There exist two rational 3-dimensional projective varieties Y, Y' such that Y is homeomorphic to Y' but the Lawson homology group $L_1H_3(Y)$ is not isomorphic to $L_1H_3(Y')$ even up to torsion.*

Proof. If $X \subset \mathbb{P}^3$ is a general smooth quartic surface, then the Picard group $\text{Pic}(X) \cong \mathbf{Z}$ by Noether-Lefschetz Theorem. For details, see e.g. Voisin [V]. But it is well known that there are still many special smooth quartic surfaces X' in \mathbb{P}^3 with $\text{rk}(\text{Pic}(X'))$ as big as 20. Note that by Theorem 3.1 and the Weak Lefschetz Theorem $L_1H_2(X) \cong \text{Pic}(X)$ for any smooth surface X in \mathbb{P}^3 .

Now we choose smooth X with $L_1H_2(X) \cong \mathbf{Z}$ and X' with $L_1H_2(X') \cong \mathbf{Z}^{20}$. Set $Y := \widetilde{\widetilde{V_3}}$ and $Y' := \widetilde{\widetilde{V'_3}}$. Let W_3 (resp. W'_3) be as in Proposition 3.1. From the proof of Lemma 2.1 in [H1], we have the commutative diagram

$$\begin{array}{ccccccccc} \cdots & \rightarrow & L_1H_3(E) & \rightarrow & L_1H_3(W_3) & \rightarrow & L_1H_3(W_3 - E) & \rightarrow & L_1H_2(E) & \rightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow \cong & & \downarrow & & \\ \cdots & \rightarrow & L_1H_3(q) & \rightarrow & L_1H_3(Y) & \rightarrow & L_1H_3(Y - q) & \rightarrow & L_1H_2(q) & \rightarrow & \cdots \end{array}$$

By Lemma 3.5, we know $E \cong X$. By Theorem 3.1, we have $L_1H_3(X) \cong H_3(X)$. By the Lefschetz Hyperplane Theorem, we know X is simply connected. Since q is a point, we have $L_1H_3(Y) \cong L_1H_3(Y - q) \cong L_1H_3(W_3 - E)$ and $L_1H_2(Y) \cong L_1H_2(Y - q) \cong L_1H_2(W_3 - E)$.

The top row of the above commutative diagram turns into the long exact sequence

$$0 \rightarrow L_1H_3(W_3) \rightarrow L_1H_3(Y) \rightarrow L_1H_2(X) \rightarrow L_1H_2(W_3) \rightarrow L_1H_2(Y) \rightarrow 0$$

Therefore, we have

$$\text{rk}L_1H_3(W_3) - \text{rk}L_1H_3(Y) + \text{rk}L_1H_2(X) - \text{rk}L_1H_2(W_3) + \text{rk}L_1H_2(Y) = 0$$

Since W_3 is a smooth rational threefold, we have $L_1H_3(W_3) \cong H_3(W_3)$, $L_1H_2(W_3) \cong H_2(W_3)$ ([FW, Prop. 6.16]) and by Proposition 3.1 $H_i(W_3) \cong H_i(W'_3)$ for all i .

$$(*) \quad \text{rk}L_1H_3(Y) = \text{rk}H_3(W_3) + \text{rk}L_1H_2(X) - \text{rk}H_2(W_3) + \text{rk}L_1H_2(Y)$$

By applying Theorem 2.1 to $(\widetilde{\widetilde{V_3}}, D_1)$, we get

$$\cdots \rightarrow L_1H_2(D_1) \rightarrow L_1H_2(\widetilde{\widetilde{V_3}}) \rightarrow L_1H_2(\widetilde{\widetilde{V_3}} - D_1) \rightarrow 0$$

Hence

$$\begin{aligned} \text{rk}L_1H_2(\widetilde{\widetilde{V_3}}) &\leq \text{rk}L_1H_2(D_1) + \text{rk}L_1H_2(\widetilde{\widetilde{V_3}} - D_1) \\ &= \text{rk}L_1H_2(D_1) + \text{rk}L_1H_2(\widetilde{\widetilde{V_3}} - q) \\ &= \text{rk}L_1H_2(D_1) + \text{rk}L_1H_2(\widetilde{\widetilde{V_3}}) \\ &\leq \text{rk}L_1H_2(D_1) + 1 \quad (\text{Lemma 3.3}) \end{aligned}$$

Similarly,

$$\text{rk}L_1H_2(\widetilde{\widetilde{V_3}}) \leq \text{rk}L_1H_2(D_2) + \text{rk}L_1H_2(\widetilde{\widetilde{V_3}})$$

Therefore,

$$\mathrm{rk} L_1 H_2(\widetilde{\widetilde{V_3}}) \leq \mathrm{rk} L_1 H_2(D_2) + \mathrm{rk} L_1 H_2(D_1) + 1.$$

Since D_1 (also D_2) is isomorphic to a $\mathbb{P}^1 \cup \mathbb{P}^1$ -bundle over a smooth curve C , it is easy to compute, by using Theorem 2.1 and the Projective Bundle Theorem [FG], that

$$\mathrm{rk} L_1 H_2(D_1) \leq \mathrm{rk} L_1 H_2(C) + 2 \cdot \mathrm{rk} L_0 H_0(C) = 1 + 2 \times 1 = 3.$$

Therefore

$$\mathrm{rk} L_1 H_2(\widetilde{\widetilde{V_3}}) \leq 3 + 3 + 1 = 7.$$

The same computation applies to $\widetilde{\widetilde{\widetilde{V'_3}}}$ and we get

$$\mathrm{rk} L_1 H_2(\widetilde{\widetilde{\widetilde{V'_3}}}) \leq 3 + 3 + 1 = 7.$$

From this together with $(*)$, we have

$$\begin{aligned} \mathrm{rk} L_1 H_3(\widetilde{\widetilde{V_3}}) &\leq \mathrm{rk} H_3(W_3) + \mathrm{rk} L_1 H_2(X) - \mathrm{rk} H_2(W_3) + 7 \\ &= \mathrm{rk} H_3(W_3) - \mathrm{rk} H_2(W_3) + 8 \quad (\text{since } L_1 H_2(X) \cong \mathbf{Z}) \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \mathrm{rk} L_1 H_3(\widetilde{\widetilde{\widetilde{V'_3}}}) &\geq \mathrm{rk} H_3(W'_3) + \mathrm{rk} L_1 H_2(X') - \mathrm{rk} H_2(W'_3) \\ &= \mathrm{rk} H_3(W_3) + \mathrm{rk} L_1 H_2(X') - \mathrm{rk} H_2(W_3) \\ &= \mathrm{rk} H_3(W_3) - \mathrm{rk} H_2(W_3) + 20 \quad (\text{since } L_1 H_2(X') \cong \mathbf{Z}^{20}) \end{aligned}$$

This shows that $L_1 H_3(\widetilde{\widetilde{V_3}})$ is not isomorphic to $L_1 H_3(\widetilde{\widetilde{\widetilde{V'_3}}})$.

□

3.2 Application to the Case $n = 3$

With this construction, if we choose $n = 3$ and $X \subset \mathbb{P}^4$ to be a general hypersurface of degree $d = 5$, then $V_3 = X - X \cap \mathbb{P}^3$ and $S := X \cap \mathbb{P}^3$ is a smooth surface in \mathbb{P}^3 .

The proof of Theorem 1.1: By applying Theorem 2.1 to the pair (X, S) , we get

$$\cdots \rightarrow L_1 H_3(V_3) \rightarrow L_1 H_2(S) \rightarrow L_1 H_2(X) \rightarrow L_1 H_2(V_3) \rightarrow 0. \quad (2)$$

The above long exact sequence (2) remains exact after being tensored with \mathbb{Q} . Note that $L_1 H_2(X) \otimes \mathbb{Q} \supset \mathrm{Griff}_1(X) \otimes \mathbb{Q}$ is an infinite dimensional \mathbb{Q} -vector space by [C]. Recall that $L_1 H_2(S)$ is finitely generated since $\dim S = 2$ (cf. [F]). Hence $L_1 H_2(V_3) \otimes \mathbb{Q}$ is

an infinite dimensional \mathbb{Q} -vector space. By (2.2), we have $L_1H_3(V_4) \otimes \mathbb{Q} \cong L_1H_2(V_3) \otimes \mathbb{Q}$ is an infinite dimensional \mathbb{Q} -vector space.

Note that $Z_3 = \overline{V_4} - V_4$ is defined by $(x_5 \cdot f(0, x_1, \dots, x_4) = 0, x_0 = 0)$ in P^5 . Let $S' = (x_5 = 0) \cap (f(0, x_1, \dots, x_4) = 0)$ in the hyperplane $(x_0 = 0) \subset P^5$. It is easy to see that $S' \cong S$. Then $Z_3 = P^3 \cup \Sigma_p(S)$, where $\Sigma_p(S)$ means the joint of S and the point $p = [1 : 0 : \dots : 0]$. By applying Theorem 2.1 to the pair $(Z_3, \Sigma S)$, we get

$$\cdots \rightarrow L_1H_3(Z_3 - \Sigma S) \rightarrow L_1H_2(\Sigma S) \rightarrow L_1H_2(Z_3) \rightarrow L_1H_2(Z_3 - \Sigma S) \rightarrow 0. \quad (3)$$

Note that $Z_3 - \Sigma S \cong P^3 - S$. Therefore $L_1H_2(Z_3) \otimes \mathbb{Q}$ is of finite dimensional since both $L_1H_2(\Sigma S) \otimes \mathbb{Q} \cong L_0H_0(S, \mathbb{Q}) \cong \mathbb{Q}$ ([L1]) and $L_1H_2(P^3 - S) \otimes \mathbb{Q}$ are. By the same type argument, we have $L_1H_3(Z_3) \otimes \mathbb{Q}$ is of finite dimensional since both $L_1H_3(\Sigma S) \otimes \mathbb{Q} \cong L_0H_1(S, \mathbb{Q}) = 0$ (note that S is simply connected) and $L_1H_3(P^3 - S) \otimes \mathbb{Q}$ are.

By applying Theorem 2.1 to the pair $(\overline{V_4}, Z_3)$, we have the following long exact sequence

$$\cdots \rightarrow L_1H_3(Z_3) \rightarrow L_1H_3(\overline{V_4}) \rightarrow L_1H_3(V_4) \rightarrow L_1H_2(Z_3) \rightarrow \cdots \quad (4)$$

From (4), the infinite dimensionality of $L_1H_3(V_3) \otimes \mathbb{Q}$, the finite dimensionality of $L_1H_2(Z_3) \otimes \mathbb{Q}$ and $L_1H_3(Z_3) \otimes \mathbb{Q}$, we obtain that $L_1H_3(\overline{V_4}) \otimes \mathbb{Q}$ is an infinitely dimensional \mathbb{Q} -vector space. This completes the proof of Theorem 1.1. \square

We can continue the procedure. Set $V_5 := \mathbb{C}^5 - V_5$, then V_5 can be viewed as an affine variety in \mathbb{C}^6 defined by $x_6 \cdot (x_5 \cdot f(1, x_1, \dots, x_4) - 1) - 1 = 0$. Set $Z_4 = \overline{V_5} - V_5$, and so on. It can be shown in the same way that $L_1H_3(Z_4)$ is finitely generated by using Theorem 2.1 and Lawson's Complex Suspension Theorem. Note that $L_1H_4(V_5) \cong L_1H_3(V_4)$ is infinitely generated by 2.2.

By applying Theorem 2.1 to the pair $(\overline{V_5}, Z_4)$, we get the long exact sequence

$$\rightarrow L_1H_4(Z_4) \rightarrow L_1H_4(\overline{V_5}) \rightarrow L_1H_4(V_5) \rightarrow L_1H_3(Z_4) \rightarrow \cdots$$

From these we obtain that $L_1H_4(\overline{V_5})$ is infinitely generated.

Proposition 3.2 *In this construction, $L_1H_k(\overline{V_{k+1}})$ is **not** finitely generated for $k \geq 3$.* \square

By the Complex Suspension Theorem [L1], $L_{p+1}H_{2p+k}(\Sigma^p \overline{V_{k+1}}) \cong L_1H_k(\overline{V_{k+1}})$. Therefore we get:

Theorem 3.3 *For integers p and k , with $k > 0, p > 0$, we can find a rational projective variety Y , such that $L_pH_{2p+k}(Y)$ is infinitely generated.* \square

Remark 3.2 If $k = 0$ and $p > 0$, there also exists projective varieties Y such that $L_p H_{2p}(Y)$ is infinitely generated. This follows from the Projective Bundle Theorem [FG] and a result of Clemens [C].

Remark 3.3 All the Y thus constructed above are singular projective varieties. Can one find some smooth projective variety such that the answer to the question (Q) is positive? Yes, we can. The author has constructed examples of smooth projective varieties such (Q) is true (cf. [H2]).

Remark 3.4 Note that all $\overline{V_{k+1}}$ are singular rational projective varieties. For smooth rational projective varieties Y , $L_1 H_*(Y) \otimes \mathbb{Q}$ are finite dimensional \mathbb{Q} -vector spaces [Pe]. The author showed $L_1 H_*(Y)$ are finitely generated abelian groups [H1].

4 Appendix

Let $f(x_0, \dots, x_4)$ be a general homogenous polynomial of degree 5 and X be a hypersurface of degree 6 in \mathbb{P}^5 given by $F(x_0, \dots, x_5) := x_5 f(x_0, \dots, x_4) - x_0^6 = 0$. It is easy to see from the proof of Lemma 3.4 that the singular points set of X is the union of a smooth 2-dimensional variety Y given by $x_0 = x_5 = f(x_0, x_1, \dots, x_4) = 0$ and an isolated point defined by $\{x_0 = \frac{\partial f}{\partial x_0} = \frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \dots = \frac{\partial f}{\partial x_5} = 0\}$.

Let $\sigma : \widetilde{\mathbb{P}}_Y^5 \rightarrow \mathbb{P}^5$ be the blow up of \mathbb{P}^5 along the surface Y and \tilde{X}_Y be the proper transform in the blow up $\widetilde{\mathbb{P}}_Y^5$. Denoted by $E = \mathbb{P}(N_{Y/\mathbb{P}^5})$ the exceptional divisor of the blow-up. Then $D = E \cap \tilde{X}_Y \subset \mathbb{P}(N_{Y/\mathbb{P}^5})$ corresponds to the image of the tangent cones $T_p X \subset T_p(\mathbb{P}^5)$ in $\mathbb{P}(N_{Y/\mathbb{P}^5})$ at points $p \in Y$.

Now

$$T_p X = \left\{ \sum_{i_0 + \dots + i_5 = 2} \frac{\partial^2 F}{\partial x_0^{i_0} \dots \partial x_5^{i_5}} x_0^{i_0} \dots x_5^{i_5} = 0 \right\}$$

a degree 2 polynomial in \mathbb{P}^5 . Directly computation shows that

$$\begin{aligned} T_p X &= \left\{ \frac{\partial f}{\partial x_0}(p) x_0 x_5 + \dots + \frac{\partial f}{\partial x_4}(p) x_4 x_5 = 0 \right\} \\ &= (x_5 = 0) \cup \left\{ \frac{\partial f}{\partial x_0}(p) x_0 + \dots + \frac{\partial f}{\partial x_4}(p) x_4 = 0 \right\}. \end{aligned}$$

Hence $D = \tilde{X}_Y \cap E$ is a fiber bundle over Y with singular conics as fibers. Clearly, \tilde{X}_Y is smooth away from D . Since $D \subset E$ is a 3-dimensional variety with singular points set $S \cong Y$, we can show that it is the only singularity on \tilde{X}_Y :

Proposition 4.1 *The proper transform \tilde{X}_Y is a 4-dimensional variety in $\mathbb{P}(N_{Y/\mathbb{P}^5})$ with singularity $S \cong Y \cup \{q\}$, where q is an isolated singular point.*

Proof. From the proof of Lemma 3.4, we see that the singular points S of X consist of two components. One is a smooth surface and the other is an isolated point q .

Since f is nonsingular on $Y = \{x_0 = f(0, x_1, \dots, x_4) = 0\}$, we have $df \neq 0$ on Y . Let us restrict ourselves to a neighborhood of a point p in Y . There, we can take the neighborhood of p as the affine space \mathbb{C}^5 with p the origin. Hence we can choose $y = f$ as a coordinate in the neighborhood of each point on Y since it is smooth. Locally, Y is defined by $x_0 = x_5 = y = 0$ in \mathbb{C}^5 . We denote it by Y_0 . For convenience, we denote x_0 by x , x_5 by z . The blow up $(\widetilde{\mathbb{C}^5})_{Y_0}$ of \mathbb{C}^5 along Y_0 is defined by the system of equations

$$\begin{cases} xv = uy, \\ xw = uz, \\ yw = zv. \end{cases}$$

in $\mathbb{C}^5 \times \mathbf{P}^2$, where $[u : v : w]$ is the homogenous coordinates on \mathbf{P}^2 . Let $\sigma : (\widetilde{\mathbb{C}^5})_{Y_0} \rightarrow \mathbb{C}^5$ be the map of this blowup. Then the inverse image of X is given by the following equations:

$$\begin{cases} x^6 - yz = 0, \\ xv = uy, \\ xw = uz, \\ yw = zv. \end{cases}$$

The above equations define two divisors on $(\widetilde{\mathbb{C}^5})_{Y_0}$. One of them is the exceptional divisor E_0 , the intersection of E with $(\widetilde{\mathbb{C}^5})_{Y_0}$ and the other is exactly the proper transform \tilde{X}_{Y_0} of X_0 in $(\widetilde{\mathbb{C}^5})_{Y_0}$, where X_0 is the part of X in \mathbb{C}^5 .

We want to show that \tilde{X}_{Y_0} is smooth away from Y_0 . Now it is clear. The blow up $(\widetilde{\mathbb{C}^5})_{Y_0}$ is covered by 3 open charts: $(u \neq 0)$, $(v \neq 0)$ and $(w \neq 0)$.

On the chart $(u \neq 0)$, we can set $u = 1$. The equations for the inverse image of X_0 under σ are given by

$$\begin{cases} x^6 - yz = 0, \\ xv = y, \\ xw = z, \\ yw = zv. \end{cases}$$

The equations $xv = y$ and $xw = z$ imply $yw = zv$. Replacing y and z by xv and xw , respectively, we can factor x^2 in the first equation $x^6 - (xv)(xw) = 0$. Hence the proper transform \tilde{X}_{Y_0} are given by

$$\begin{cases} x^4 - vw = 0, \\ xv = y, \\ xw = z. \end{cases}$$

and the exceptional divisor is given by

$$\begin{cases} x^2 = 0, \\ xv = y, \\ xw = z. \end{cases}$$

i.e., $x = y = z = v = w = 0$, which is isomorphic to Y_0 .

It is easy to show that on the charts $(v \neq 0)$ and $(w \neq 0)$, the proper transform \tilde{X}_{Y_0} is smooth everywhere. This completes the proof of the proposition. \square

Remark 4.1 *In fact, the 2-dimensional singularity of X is of the A_n -type. It can be resolved by blowing up one more time. The isolated singularity q can be resolved by one blowup.*

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Reference

- [C] Clemens, H., *Homological equivalence, modulo algebraic equivalence, is not finitely generated*, I.H.E.S. Publ. Math. 58 (1983), 19-38.
- [DT] Dold, A. and Thom, R., *Quasifaserungen und unendliche symmetrische Produkte*. (German) Ann. of Math. (2) 67 1958 239–281.
- [F] Friedlander, E. *Algebraic cycles, Chow varieties, and Lawson homology*, Comp. Math. 77(1991), 55-93.
- [FG] Friedlander, Eric M.; Gabber, Ofer, *Cycle spaces and intersection theory*, Topological methods in modern mathematics (Stony Brook, NY, 1991), 325–370, Publish or Perish, Houston, TX, 1993.
- [H1] Hu, W., *Some birational invariants defined by Lawson homology*. arXiv:math.AG/0511722.
- [H2] Hu, W., *Generalized Abel-Jacobi map on Lawson homology*. Preprint.
- [L1] Lawson, H.B. Jr., *Algebraic cycles and homotopy theory*, Ann. of Math. **129**(1989), 253-291.
- [L2] Lawson, H.B. Jr., *Spaces of algebraic cycles*, pp. 137-213 in Surveys in Differential Geometry, 1995 vol.2, International Press, 1995.
- [Li] Lima-Filho, P., *Lawson homology for quasiprojective varieties*, Comp. Math. 84 (1992), no. 1, 1–23.

- [Pe] Peters, C. *Lawson homology for varieties with small Chow groups and the induced filtration on the Griffiths groups*, Math. Z. 234 (2000), no. 2, 209–223.
- [V] Voisin, C., *Théorie de Hodge et géométrie algébrique complexe*, Cours Spécialisés, 10. Société Mathématique de France, Paris, 2002. viii+595 pp. ISBN 2-85629-129-5

Department of Mathematics,
Stony Brook University, SUNY,
Stony Brook, NY 11794-3651
Email:wenchuan@math.sunysb.edu